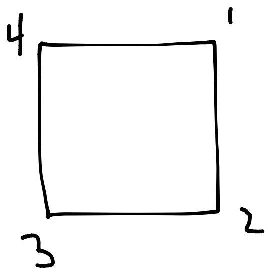
Dihedral Groups

Goal: Study symmetries of an n-gon.

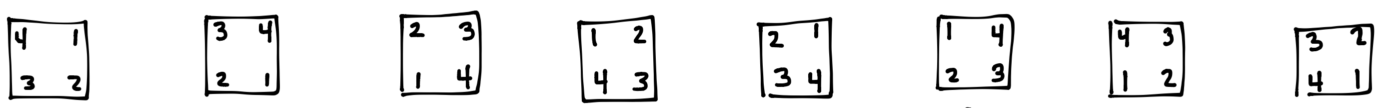
Example: D_8 (symmetries of a square)

Let D_8 be the set of symmetries of a square.
A symmetry is a rigid motion where the square is replaced so that it exactly covers its original position.

We start w/ the square w/ its vertices labelled:



we can then replace the square in 8 different ways:



e (or 1)	r	r^2	r^3	s	sr	sr^2	sr^3
leave the square alone	rotate clockwise $90°$	rotate clockwise $180°$	rotate clockwise $270°$	flip over	flip over (or do r then s)	flip over (or do s then r^2)	flip over (or do r^3 then s)

Note that the actions are performed from right to left.
This is because we are thinking of these as functions on the vertices of the square.

In fact, because our operation is composition of functions, we know it's associative. We will soon see why it satisfies the other two axioms.

Here is the multiplication table:

	1	r	r^2	r^3	s	sr	sr^2	sr^3
1	1	r	r^2	r^3	s	sr	sr^2	sr^3
r	r	r^2	r^3	1	sr^3	s	sr	sr^2
r^2	r^2	r^3	1	r	sr^2	sr^3	s	sr
r^3	r^3	1	r	r^2	sr	sr^2	sr^3	s
s	s	sr	sr^2	sr^3	1	r	r^2	r^3
sr	sr	sr^2	sr^3	s	r^3	1	r	r^2
sr^2	sr^2	sr^3	s	sr	r^2	r^3	1	r
sr^3	sr^3	s	sr	sr^2	r	r^2	r^3	1

s row, r column $= sr =$ "do r then s"

1.) Check that 1 is in fact the identity.

2.) Does every element have an inverse? What is it? i.e. what is $(s^i r^j)^{-1}$?

3.) Is the group abelian?

4.) Notice that every element can be written as $r^i s^j$ for $0 \leq i \leq 3$, $0 \leq j \leq i$. What is the "rule" for writing any element this way? i.e. what is $s^m r^n$?

5.) What is the order of each element?

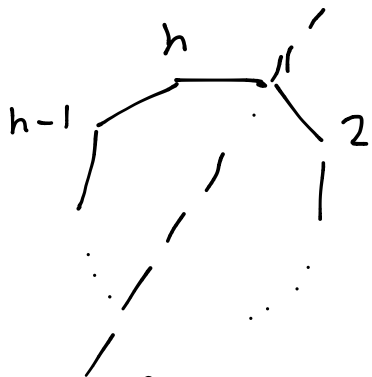
Note that every element can be written in terms of s and r. These are called generators:

Def: Let G be a group. $S \subseteq G$ is a set of generators of G is every element of G can be written as a finite product of elements of S and their inverses. Then G is generated by S.

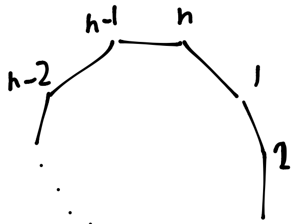
Dihedral groups in general

D_{2n} is the group of symmetries of an n-gon.

Again, we can generate all the elements by a rotation and a flip.



e

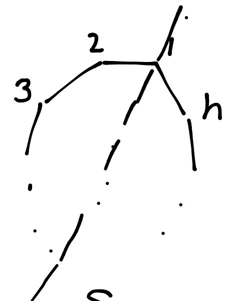


r

rotate clockwise by $\left(\frac{360}{n}\right)^\circ$

$r(1)=2$

(The 1st position goes to second position)



s

reflect over line $y=x$, where n-gon is centered at the origin

$s(1)=1$, $s(2)=n$, $s(3)=n-1, \dots$

Properties of D_{2n}

1.) $1, r, r^2, \dots, r^{n-1}$ are all distinct and $r^n=1$, so $|r|=n$

2.) $s^2=1$

3.) $s \neq r^i$ for any i.

4) $sr^i \neq sr^j$ for $i \neq j$ and $i,j \in \{0,1,\dots,n-1\}$

i.e. each element can be written uniquely as $s^m r^n$ for $m \in \{0,1\}$, $n \in \{0,\dots,n-1\}$

5) $rs = sr^{-1}$ $(= sr^{n-1})$.

Thus, s and r don't commute (unless $n=2$), so D_{2n} is not abelian.

6.) $r^i s = sr^{-i}$ $(= sr^{n-i})$ $\forall$ $0 \leq i \leq n-1$.

Pf of 3: r^i only fixes 1 if $r^i = 1$. s fixes 1, so $s \neq r^i$. $\square$

Pf of 4: $sr^i(n-i+1) = 1 = sr^j(n-j+1)$. So $i \neq j \Rightarrow sr^i \neq sr^j$, since each elt is a bijection on the vertices.

Pf of 5: If $rs = r^i$, then $r^{n-1} rs = r^{n+i-1} \Rightarrow s = r^{n+i-1}$, which contradicts 3.).

Thus $rs = sr^i$ for some $i \in \{0,\dots,n-1\}$

$s(1) = 1$ and $r(1) = 2$, so $rs(1) = 2$.

[Think: R sends whatever's here to here]

Thus, $2 = sr^i(1) \Rightarrow r^i(1) = n \Rightarrow i = n-1$. Thus $rs = sr^{n-1} = sr^{-1}$! $\square$

(s sends whatever is in the n^{th} spot to the 2^{nd} spot)

Pf of 6: We prove $r^i s = sr^{-i}$ by induction on i.

5.) gives the base case

Then $r^i s = r(r^{i-1}s) = r(sr^{-i+1}) = (sr^{-1})r^{-i+1} = sr^{-i}$. $\square$

Note that for each n, the generators of D_{2n} are r, s, and we've shown they satisfy $r^n = 1$, $s^2 = 1$, and $rs = sr^{-1}$. These are called <u>relations</u>.

In fact, <u>any</u> other equation involving the generators can be derived from these.

Any such collection of generators S and relations $R_1, \ldots, R_m$ for a group G is called a <u>presentation</u>, written

$$G = \langle S \mid R_1, \ldots, R_m \rangle$$

So, e.g. $D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle$.

Note that a presentation is not unique!